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Continuous percolation in one dimension

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Abstract. The continuous percolation problem for equal disc size in a one-dimensional space is treated. Exact expressions are given for the relevant functions and the critical exponents. It is shown that with an appropriate selection of scaling fields, this problem, as well as the further-neighbour discrete problem, belongs to the same universality class as the more common nearest-neighbour one-dimensional discrete problem.

1. Introduction

The percolation problem of discrete lattices has been intensively studied in recent years. Although many phenomena can be better simulated by continuous models (e.g. polymerisation and gelation), they were mostly approximated by a discrete problem, which is indeed much easier to manipulate. Only relatively few papers (e.g. Pike and Seager 1974, Fremlin 1976, Haan and Zwanzig 1977, Wintle and Puhach 1978, Powell 1979, 1980, Vicsek and Kertesz 1981) on continuous percolation have been published. The present note is a treatment of the simplest continuous percolation problem—the one-dimensional one. Like many other one-dimensional problems, the percolation problem may also be exactly solved. It is hoped that the solution will lead to some insight in higher dimensionality. The results here are not surprising, but still they are not devoid of some interest and are worth some discussion.

Let us define the problem. The infinite one-dimensional space $-\infty < x < \infty$ is randomly covered by 'discs' of length d with uniform probability density ρ, i.e. on average there are $\rho\,\mathrm{d}x$ centres of discs on an interval of length $\mathrm{d}x$. A sequence of discs such that any two nearest neighbours overlap belong to the same cluster. Clusters vary by both s, the number of discs, and x, their size—the maximum distance between any two ends of discs of the cluster. It is evident that for finite clusters $s \geqslant 1$ the following relation exists,

$$1 \leqslant x \leqslant sd, \tag{1}$$

where the equality holds on both sides only for $s = 1$, otherwise the strict inequality holds with probability 1. The system percolates if there is a cluster with an infinite number of discs, s, or equivalently, with infinite length x. (A more complicated problem may be defined with discs of different sizes. Though more complicated it is still manageable.)

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2. Bi-connectedness function and cluster statistics

The distribution of the disc centres (or their left or right ends) is naturally a Poisson distribution. The probability $\bar{P}(s, a)$ to have exactly s centres, not necessarily belonging to the same cluster, in the interval $[0, a]$ is

$$\bar{P}(s, a) = (\rho a)^s \, \mathrm{e}^{-\rho a}/s!, \tag{2}$$

and in particular, for $s = 0$ we obtain

$$\bar{P}(0, a) = \mathrm{e}^{-\rho a} \tag{2a}$$

and

$$\sum_{s=0}^{\infty} \bar{P}(s, a) = 1. \tag{2b}$$

A more interesting and relevant quantity is the bi-connectedness function $c_2(0, x)$, which is defined here as the probability density that given a disc centre at the origin, there is a disc, belonging to the same cluster, the centre of which is at x. (This is not the only possible and useful definition: e.g. one can substitute 'the centre of which is at x' by 'which covers x'.) Evidently

$$c_2(0, x) = \sum_{s=1}^{\infty} c_2^{(s)}(0, x) \equiv \sum_{s=1}^{\infty} c_2^{(s)}(x_0 = 0, x_s = x) \tag{3}$$

where $c_2^{(s)}(0, x)$ is the probability density that between 0 and x there are exactly $s+1$ discs (including the two end discs) belonging to the same cluster. Let us calculate the individual $c_2^{(s)}(0, x)$, assuming $x > 0$. The probability density that given $x_0 = 0$, the centres of the next s discs to the right belong to the same cluster, and their centres are at $0 < x_1 < x_2 < \ldots < x_s = x$, is

$$\rho^s \, \mathrm{e}^{-\rho x} f(x_1) f(x_2 - x_1) \ldots f(x_s - x_{s-1}) \tag{4}$$

where $f(x)$ is the overlapping function

$$\begin{aligned} f(x) &= 1, \qquad 0 \leq x \leq d, \\ f(x) &= 0, \qquad \text{otherwise}. \end{aligned} \tag{5}$$

Hence

$$\begin{aligned} c_2^{(s)}(0, x) &= \rho^s \, \mathrm{e}^{-\rho x} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1) f(x_2 - x_1) \ldots f(x_s - x_{s-1}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_{s-1} \\ &\equiv \rho^s \, \mathrm{e}^{-\rho x} F(s, x). \end{aligned} \tag{6}$$

$F(s, x)$ is just the $(s-1)$ multiple convolution integral in (6). It is worked out in standard probability textbooks (e.g. Dwass 1976), but let us work it out again briefly. $\phi(t)$, the Fourier transform of $f(x)$, satisfies

$$\phi(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}tx} f(x) \, \mathrm{d}x = (\mathrm{e}^{\mathrm{i}td} - 1)/\mathrm{i}t. \tag{7}$$

Hence the Fourier transform of $F(x)$ is $\Phi(t) = (\phi(t))^s$ so that

$$F(s, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt} [(\mathrm{e}^{\mathrm{i}td} - 1)/\mathrm{i}t]^s \, \mathrm{d}t. \tag{8}$$

Differentiating (8) s times with respect to x, one obtains

$$\begin{aligned}\frac{\mathrm{d}^s}{\mathrm{d}x^s}F(s,x)&=\frac{(-1)^s}{2\pi}\int \mathrm{e}^{-\mathrm{i}xt}(\mathrm{e}^{\mathrm{i}td}-1)^s\,\mathrm{d}t\\ &=\frac{(-1)^s}{2\pi}\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}\int \mathrm{e}^{\mathrm{i}t(kd-x)}\,\mathrm{d}t\\ &=\sum_{k=0}^{\infty}\binom{s}{k}(-1)^k\,\delta(x-kd).\end{aligned} \tag{9}$$

Integrating (9) s times results in

$$F(s,x)=\sum_{k=0}^{s}(-1)^k\binom{s}{k}(x-kd)^{s-1}\theta(x-kd)/(s-1)! \tag{10}$$

where θ is the Heaviside function, and specifically

$$\theta(x-kd)=\int_{-\infty}^{x}\delta(x'-kd)\,\mathrm{d}x'=\begin{cases}0 & \text{for } x<kd,\\ 1 & \text{for } x\geqslant kd.\end{cases} \tag{11}$$

It is obvious by construction that $F(s,x)=0$ for $x<0$ or $x>sd$ and that it is symmetric around $x=sd/2$: $F(sd/2-y)=F(sd/2+y)$. Some graphs of $F(s,x)$ are given for several s in figure 1. For large s, $F(s,x)$ tends to a gaussian function:

$$\begin{aligned}\phi(t)&=d\,\mathrm{e}^{\mathrm{i}dt/2}(\sin dt/2)/(dt/2)=d\,\mathrm{e}^{\mathrm{i}dt/2}[1-(dt)^2/24+(dt)^4/1920\ldots]\\ &=d\exp(\mathrm{i}dt/2)\exp[-(dt)^2/24]\exp[-(dt)^4/2880]\ldots\end{aligned}$$

and

$$(\phi(t))^s\approx d^s\,\mathrm{e}^{\mathrm{i}sdt/2}\,\mathrm{e}^{-s(dt)/24}. \tag{12}$$

The higher exponents may be ignored in the first approximation. The inverse Fourier transform yields

$$F(s,x)\approx d^{s-1}(6/\pi s)^{1/2}\exp[-6(x-sd/2)^2/sd^2]=F^{\mathrm{g}}(s,x). \tag{13}$$

For $F^{\mathrm{g}}(s,x)$, the gaussian approximation, the first two moments are the correct ones for all s:

$$\langle x\rangle=sd/2,\qquad \langle x^2\rangle-\langle x\rangle^2=sd^2/12. \tag{14}$$

There is a slight discrepancy in the fourth moment,

$$\langle x-\langle x\rangle\rangle^4_{\mathrm{exact}}=(s^2/48-s/120)d^4,\qquad \langle x-\langle x\rangle\rangle^4_{\mathrm{gaussian}}=s^2d^4/48, \tag{15}$$

which indicates that the approximation is a very good one already for moderately low s. It is obvious that $F^{\mathrm{g}}(s,x)$ does not exactly vanish outside the interval $[0,s]$, but this has but a minor effect for not too small s. These facts are well reflected in figure 1.

With equations (3), (6) and (10) $c_2(0,x)$ has an explicit expression, which after some manipulation acquires the form

$$\begin{aligned}c_2(0,x)&=\sum_{s=1}^{\infty}\rho^s\,\mathrm{e}^{-x\rho}\sum_{k=0}(-1)^k\binom{s}{k}(|x|-kd)^{s-1}\theta(|x|-kd)/(s-1)!\\ &=\rho\sum_{k=0}^{\infty}(-1)^k\theta(|x|-kd)\left(\frac{[\mathrm{e}^{-\rho d}\rho(|x|-kd)]^{k-1}}{(k-1)!}+\mathrm{e}^{-\rho d}\frac{[\mathrm{e}^{-\rho d}\rho(|x|-kd)]^k}{k!}\right).\end{aligned} \tag{16}$$

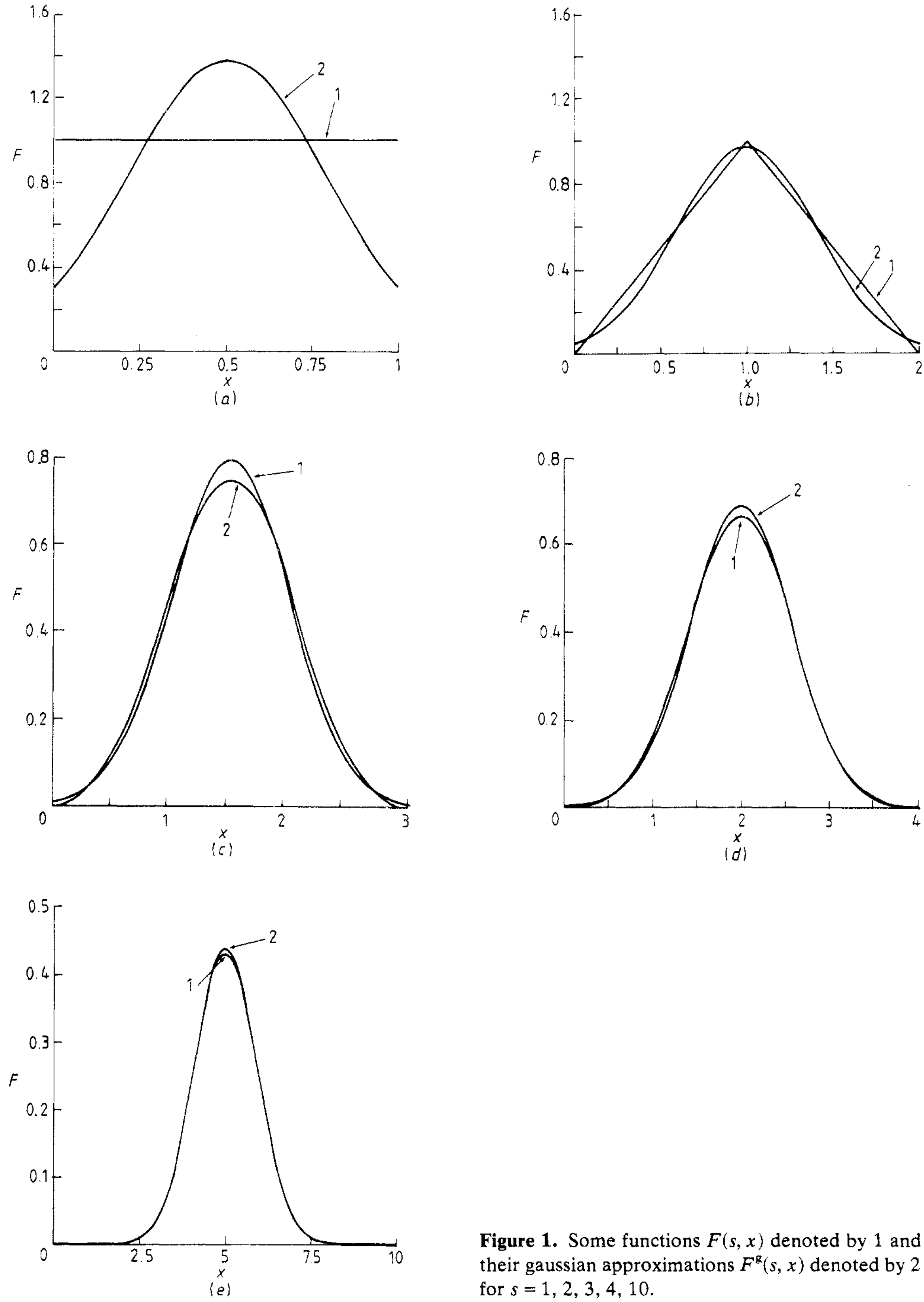

Figure 1. Some functions $F(s, x)$ denoted by 1 and their gaussian approximations $F^{g}(s, x)$ denoted by 2 for $s=1, 2, 3, 4, 10$.

This expression is not very transparent. It may estimated for large x, by substituting $x-kd$ by x and thus having

$$c_2(0, x) \approx \rho(1-\mathrm{e}^{-\rho d}) \exp(-\rho x\, \mathrm{e}^{-\rho d}), \tag{17}$$

from which the correlation length is determined:

$$\xi = e^{\rho d}/\rho. \tag{17a}$$

A more direct, but tedious, calculation of moments yields

$$\int_{-\infty}^{\infty} c_2(0, x)\, dx = 2(e^{\rho d} - 1), \tag{18}$$

$$\int_{-\infty}^{\infty} x^2 c_2(0, x)\, dx = 2\rho^{-2} e^{\rho d}[2 e^{2\rho d} - (2 + 4\rho d) e^{\rho d} + 2\rho d + (\rho d)^2]. \tag{18a}$$

Hence the expression of ξ, given by $\xi^2 = \int x^2 c_2\, dx / \int c_2\, dx$, is immediately given and especially for $\rho d \to \infty$:

$$\xi = \sqrt{2}\, e^{\rho d}/\rho. \tag{19}$$

With the aid of $c_2^{(s)}(0, x)$ some other commonly used functions in percolation theory may be obtained.

The probability density $P(x_0 = 0, x_s = x)$ of a cluster containing exactly $s + 1$ discs such that $x_0 = 0$, $x_s = x$ is obviously

$$\begin{aligned} P(x_0 = 0, x_s = x) &= \rho\, e^{-2\rho d} c_2^{(s)}(0, x) \\ &= \rho^{s+1} e^{-(x+2d)\rho} \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} \frac{(x - kd)^{s-1}}{(s-1)!} \theta(x - kd). \end{aligned} \tag{20}$$

(The additional factor $\rho\, e^{-2\rho d}$ with respect to $c_2^{(s)}(0, x)$ is due to the fact that here the occupation of the origin is not assumed, and on the other hand it is demanded that the intervals $(-d, 0)$ and $(x, x + d)$ are unoccupied.) The probability density of having an $(s + 1)$ cluster of end-to-end size $x + d$, and covering a given point (e.g. the origin), is

$$P(s+1, x+d; \rho) = \rho^{s+1} e^{-(x+2d)\rho}(x + d) \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} (x - kd)^{s-1} \theta(x - kd)/(s-1)!. \tag{21}$$

It is obvious that

$$P(s, x, \rho) = 0 \qquad \text{if} \qquad x \geqslant sd, \tag{21a}$$

$$P(1, x + d, \rho) = \rho\, e^{-2\rho d}\, \delta(x - d). \tag{21b}$$

Therefore we obtain the probability $P(s + 1, \rho)$, that the origin is covered by an $(s + 1)$ cluster whose length is immaterial. For $s \geqslant 0$

$$\begin{aligned} P(s+1, \rho) &= \int_0^{\infty} P(s+1, x+d, \rho)\, dx \\ &= e^{-2\rho d} \{s + \rho d - [s + \rho d(s+1)] e^{-\rho d}\}(1 - e^{-\rho d})^{s-1}. \end{aligned} \tag{22}$$

Specifically for $s = 0$ we have

$$P(1, \rho) = \rho d\, e^{-2\rho d}. \tag{22a}$$

It is easily found that $P(0, \rho)$, the probability that a given point is not covered, is according to $(2a)$

$$P(0, \rho) = e^{-\rho d}, \tag{22b}$$

so that

$$\sum_{s=0}^{\infty} P(s, \rho) = 1 \tag{22c}$$

is satisfied.

Similarly, the probability density $P(x+d, \rho)$ of having a cluster of length $(x+d)$ covering the origin is

$$\begin{aligned} P(x+d, \rho) &= \rho \, \mathrm{e}^{-2\rho d} c_2(0, x) = \sum_{s=0}^{\infty} P(s+1, x+d, \rho) \\ &= (x+d)\rho^2 \, \mathrm{e}^{-2\rho d} \sum_{k=1}^{\infty} (-1)^k \left(\frac{[(x-kd)\rho \, \mathrm{e}^{-\rho d}]^{k-1}}{(k-1)!} \mathrm{e}^{-\rho d} \right. \\ &\quad \left. + \frac{[(x-kd)\rho \, \mathrm{e}^{-\rho d}]^k}{k!} \right) \theta(x-kd). \end{aligned} \tag{23}$$

Some of the moments of the clusters distribution are of interest:

$$\langle s \rangle_\rho = \sum (s+1) P(s+1, \rho) = 2 \, \mathrm{e}^{\rho d} - 2 - \rho d. \tag{24}$$

At the limit $\rho \to 0$ one obtains $\langle s \rangle_\rho = \rho d$,

$$\langle s^2 \rangle_\rho = 6 \, \mathrm{e}^{2\rho d} - 8 \, \mathrm{e}^{\rho d} + 2 - 4\rho d \, \mathrm{e}^{\rho d} + \rho d, \tag{25}$$

and at the limit $\rho \to 0$, $\langle s^2 \rangle_\rho = \rho d + 4(\rho d)^2$. The standard deviation $(\Delta s)_\rho$ satisfies

$$(\Delta s)_\rho^2 = \langle s^2 \rangle_\rho - \langle s \rangle_\rho^2 = 2 \, \mathrm{e}^{2\rho d} - 3\rho d - (\rho d)^2 - 2 \tag{26}$$

and

$$\lim_{\rho \to 0} (\Delta s)_\rho = \rho d + 3(\rho d)^2.$$

With the above results percolation 'thermodynamics' can be defined.

3. Critical indices

A fundamental concept in discrete percolation theory is $\langle n_s \rangle$, the average number of s-site clusters per lattice site. Similarly, a plausible definition for the continuous case is the average number of $(s+1)$-site clusters per unit length. By equation (20) we obtain

$$\langle n(s+1) \rangle = \int \mathrm{d}x \, P(x_0 = 0, x_s = x) = \rho \, \mathrm{e}^{-2\rho d} (1 - \mathrm{e}^{-\rho d})^s. \tag{27}$$

Introducing ghost fields (Kasteleyn and Fortuin 1969, Reynolds *et al* 1977), the modified average number is

$$\langle n(s+1, \rho, h) \rangle = \rho \, \mathrm{e}^{-2\rho d} (1 - \mathrm{e}^{-\rho d})^s (1-h)^{s+1}. \tag{28}$$

The corresponding Gibbs potential is

$$G(\rho, h) = \sum_{s=0}^{\infty} \langle n(s+1, \rho, h) \rangle = \frac{\rho(1-h) \, \mathrm{e}^{-2\rho d}}{1 - (1 - \mathrm{e}^{-\rho d})(1-h)}. \tag{29}$$

The probability $P(\rho, h)$ that an occupied point belongs to an infinite cluster is

$$P(\rho, h) = 1 - e^{-2\rho d} \sum_{s=0}^{\infty} s(1 - e^{-\rho d})^{s-1}(1-h)^s = 1 - \frac{(1-h)\, e^{-2\rho d}}{[1-(1-e^{-\rho d})(1-h)]^2}. \tag{30}$$

Finally, the average site number on a finite cluster is

$$S(\rho, h) = \frac{\Sigma\, s^2\langle n(s, h)\rangle}{\Sigma\, s\langle n(s, h)\rangle} = \frac{1+(1-h)(1-e^{-\rho d})}{1-(1-h)(1-e^{-\rho d})}. \tag{31}$$

The percolation threshold ρ_c is the density where the above functions are singular. It turns out that they are regular for any finite ρ. A singularity is manifested only at $h = 0$, $\rho \to \infty$, as expected. At this limit, the thermodynamic functions are to the first non-vanishing order:

$$\begin{aligned} &G(\rho, h) \to \rho\, e^{-2\rho d}/(e^{-\rho d} + h), \qquad G(\rho, 0) \to \rho\, e^{-\rho d}, \\ &P(\rho, h) \to h(h + 2\, e^{-2\rho d})/(e^{-\rho d} + h)^2, \qquad P(\rho, 0) = 0, \qquad P(\rho, h \neq 0) \to 1, \\ &S(\rho, h) \to 2/(e^{-\rho d} + h), \qquad S(\rho, 0) \to 2\, e^{\rho d}. \end{aligned} \tag{32}$$

These functions are *not* homogeneous, or even generalised homogeneous by the strict definition. Nevertheless, critical exponents may be defined (e.g. $\gamma = \lim_{\rho\to\infty} \ln S(\rho, 0)/\ln \rho = \infty$), and the results are

$$\alpha_p = \infty, \qquad \beta_p = 0, \qquad \gamma_p = \infty, \qquad \delta_p = \infty. \tag{33}$$

These results convey but little information. A better way to treat the problem is to homogenise the functions by a transformation $r = e^{-\rho d}$, and also define $\langle n_s\rangle$ as the average number of clusters per disc rather than per unit length. Thus we have

$$\begin{aligned} &G(r, h) = r^2/(r + h), \\ &P(r, h) = h(h + 2r)/(r + h)^2, \qquad P(0, h) = 1, \\ &S(r, h) = 2/(r + h). \end{aligned} \tag{34}$$

The critical value for the new scaling field is $r_c = 0$. With this field, the newly defined critical exponents are

$$\alpha = 1, \qquad \beta = 0, \qquad \gamma = 1, \qquad \delta = \infty. \tag{35}$$

They are defined by $\varepsilon = r - r_c = r$ and

$$G(r, 0) \approx \varepsilon^{2-\alpha}, \qquad P(r, 0) \approx (-\varepsilon)^{\beta}, \qquad S(r, 0) \approx \varepsilon^{-\gamma}, \qquad P(0, h) \approx h^{1/\delta}.$$

These results coincide with the well known results for the discrete one-dimensional problem, so that the continuous problem belongs to the same universality class.

Let us also define the correlation length exponents. Equations (17) and (19) transform to

$$c_2(0, x, r) = (|\ln r|/d)\, e^{-r|\ln r|x/d}(1 - r), \qquad \xi = \sqrt{2}\, r^{-1} d/|\ln r|, \tag{36}$$

from which we obtain

$$\nu = 1, \qquad \eta = 1, \tag{37}$$

where $c_2(0, x, r) \approx e^{-x/\xi}/r^{D+\eta-2}$, $\xi \approx r^{-\nu}$ and D is the dimensionality. These results are obtained again by generalising $\nu = -\lim\, (\ln \xi)/(\ln r)$. Alternatively ξ and x may be measured by 'mean distance between disc centres', $1/\rho$, so that the expression of ξ would also be homogeneous in r:

$$\xi = \sqrt{2}\, r^{-1}. \qquad (36')$$

ξ (of equation (36) or (36′)) and G (of equation (32) or (34) respectively) satisfy the condition that $G\xi^D$ is constant, which is consistent with Aharony (1980).

The behaviour of $\langle n(s)\rangle$ at $\varepsilon = r \ll 1$, $h = 0$ is given by properly transforming equation (27),

$$\langle n(s)\rangle/\rho = r^2(1-r)^s \approx r^2\, \mathrm{e}^{-rs} = s^{-2}(rs)^2\, \mathrm{e}^{-rs}, \qquad (38)$$

which agrees with the scaling form suggested by Stauffer (1975a, b, 1979),

$$\langle n(s)\rangle \approx s^{-\tau} f(\varepsilon s^{\sigma}) \qquad \text{with} \qquad \tau = 2 \qquad \text{and} \qquad \sigma = 1, \qquad (39)$$

as has already been obtained for the discrete lattice (Reynolds *et al* 1977).

The choice of $\mathrm{e}^{-\rho d}$ as the scaling field is to some extent arbitrary. But a similar choice has already been made in the one-dimensional Ising problem (e^{-K} is chosen as the scaling field). The critical indices are, of course, influenced by this arbitrariness and may all have been multiplied by a constant, had another scaling function been chosen. This is peculiar to any transformation of the field which is not linear at the critical point. A very similar arbitrariness was pointed out recently by Klein *et al* (1978), treating the one-dimensional discrete problem with further-neighbour bonds. Their results for γ, $2-\alpha$ and ν were the universal ones multiplied by L (the number of interacting neighbours from each side). These values are obtained when the scaling field is p, the occupation probability. In fact, an equally good choice would be $q = 1-p$ in the nearest-neighbour problem, q^L in the further-neighbours bond problem and $\mathrm{e}^{-\rho d}$ in the continuous problem—all are the respective probabilities that an occupied site is at the (right) end of a cluster, and thus universality is preserved and the indices are independent of L.

It is intuitively obvious that the continuous problem is obtainable as a limiting case of the further-neighbour bond problem with $L \to \infty$, but let us derive it mathematically.

Let us begin with the common lattice with the elementary cell of size d and sites at the lattice points, and with nearest-neighbour interactions of range d. Let us divide each cell into L subcells, with an occupiable site at each subcell end, and still preserve the interaction range d. This is the L further-neighbour problem. A result which is not explicitly written in Klein *et al* (1978), but is evident there, is that for $h = 0$

$$\langle n_s\rangle = q^{2L} p (1-q^L)^{s-1}. \qquad (40)$$

In the limit $L \to \infty$, $p = \rho d/L$ with fixed ρ, we obtain $q^L = (1-\rho d/L)^L \to \mathrm{e}^{-\rho d}$ and

$$\langle n_s\rangle = (\rho d/L)\, \mathrm{e}^{-2\rho d}(1-\mathrm{e}^{-\rho d})^{s-1}. \qquad (41)$$

This result coincides effectively with our equation (27).

The discrepancy by a factor of d/L is a result of different definitions of $\langle n_s\rangle$: as number per site in (41) and per unit length in (29). There is however one exception to this analogy—for the singular case $\rho \to \infty$. Here there are two limiting processes, $\rho \to \infty$ and $L \to \infty$, which are not exchangeable, and may cause some difficulties in pursuing some problems, such as the space points renormalisation group. This method was successfully applied (Reynolds *et al* 1980) also to the L further-neighbour bond problem. However, it comes out that the result for ν is ambiguous when this method is applied to the continuous problem. This may have been anticipated, because already for large L in the discrete problem the convergence of the results is slow. This kind of difficulty is, of course, peculiar to the one-dimensional problem with the singular critical values $\rho \to \infty$ or $r \to 0$. For the more interesting problems of higher dimensionality no such difficulties are expected, and the large-cell renormalisation group method may be fruitful there.

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Note added in proof. A very recent work by E T Gawlinski and H E Stanley (1981), private communication should be added to the list of papers concerning continuous percolation. It seems to contain very good estimates of critical values for the two-dimensional system.

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